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Optimal Labelling Problems, their Relaxation and Equivalent Transformations

Рассмотрена оптимизационная задача разметок, которая есть обобщением известной задачи о совместимости ограничений, и ее размытая модификация. Описаны два подхода к поиску оптимальной размытой разметки, их достоинства и недостатки. Предложены направления дальнейших исследований.

The optimal labeling problem is considered, which is a generalization of the known Constraint Satisfaction Problem, and its relaxed simplification. Two approaches for the relaxed labeling optimization are described as well as their advantages and shortcomings. A direction of future researches is suggested.

Розглянуто оптимізаційну задачу розміток, що узагальнює відому задачу про сумісність обмежень, та її розмиту модифікацію. Описано два підходи до пошуку оптимальної розмитої розмітки, їх переваги і недоліки. Наведено напрями подальших досліджень.

Preface

An optimal labeling problem, which this paper is devoted to, is a research field where the applied problems of visual analysis and some other intelligent technologies come directly in contact with fundamental problems of a computer science. On the one hand, image segmentation [1–5], stereovision [1, 7–9] and other image processing problems [10–14] as well as speech recognition [17] can be presented in a natural way as various labeling problems. As it was mentioned in [15–16], the labeling problems arise in text analysis [17] and intellectual data bases [18] as well. On the other hand, the optimal labeling problem is a natural generalization of constraint satisfaction problem that belongs to the main research stream in modern computer science [15, 20–21]. Several questions, which arise at the junction of optimal labeling problems and visual analysis, are considered in this paper.

The paper consists of three logical parts. First of all, a simplest example of image processing is considered, which is frequently fulfilled at the very beginning of image processing technological chain. An exact formulation of the problem, which adequately expresses an applied content, results in NP-complete problem class even in this simplest case. So, it deserves and requires the most careful research on the highest scientific level. The problem class, which turns out to be an appropriate formalization for many applications, is known as an optimal labeling problem. In mathematical papers the problem is called VCSP (Valued Constrained Satisfaction Problem) with an emphasis of the fact that it is a generalization of well-known CSP (Constrained Satisfaction Problem).

Then the concept of the relaxed labeling is defined as well as a problem of the relaxed labeling optimization. The set of these problems does not form a NP-complete class and, consequently, it is not hopeless to solve them exactly in general form. However, the hopes has not justified yet. It means that no practically useful algorithm is known now, which could solve all problems of this class. Two approaches to cope with the problem are described, each having virtues and shortcomings.

Finally, several ideas are considered, which, hopefully, will result in algorithms free of shortcomings of the known algorithms.

1. The simplest example of image processing

Let $T = \{(i,j) \mid 1 \le i \le m;\ 1 \le j \le n\}$ be a rectangular region of two-dimensional integer grid called a vision field. Elements of this set are called pixels. Let $\bar{k} : T \to \{0,1\}$ be a function called an ideal image. An ideal image is not available for direct observation. Another, distorted, image $\bar{x} : T \to R$ is available. It is called a real image. Let us suppose that it is known how a real image $\bar{x}$ depends on ideal image $\bar{k}$. It means that conditional probabilities $p(\bar{x}/\bar{k})$ are given so that

$$p(\bar{x}/\bar{k}) = \prod_{t \in T} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x(t)-k(t))^2}. \qquad (1)$$

On the base of these data a strategy $q : X^T \to K^T$ has to be constructed that for each real image $\bar{x}$

[*] **Key words**: labeling, relaxed labeling, permuted supermodular optimization, belief propagation.

makes a maximal likelihood estimation of an ideal image

$$\bar{k}^* = \arg\max_{\bar{k}\in K^T} \prod_{t\in T} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x(t)-k(t))^2}. \qquad (2)$$

The applied content of the problem consists in a reasonable restoration of an image after its distortion. In the considered simplest case it is a mere two-level image quantization known as image binarization. The problem has an evident solution

$$\begin{aligned} \bar{k}^* &= 1 \ \text{ if } \ x(t) \geq 0{,}5, \\ &= 0 \ \text{ if } \ x(t) < 0{,}5. \end{aligned} \qquad (3)$$

However, a very first testing of this solution shows immediately that it differs essentially from what one would like to obtain. Too many points are restored as white ones when it is evident for a human observer that they are black and vice versa. That is why various post-processing procedures are used after solution (3) that have to improve the obtained results. These improvements are based on such or another reasonable considerations, which are known to a user but were not taken into account in the formal requirement (2). They are intuitive reasons that an equality $k(t)=k(t')$ for two neighboring pixels t and t' of an ideal image is more probable then inequality $k(t)\neq k(t')$. If this intuitive reasoning would be unambiguously and exactly formulated, the problem (2) could be modified so that its solution would require no additional improvement. For example, one can assume that a priori probability distribution $p: K^T \to R$ is defined on the set K^T of all possible ideal images so that a priori probability $p(\bar{k})$ of an image $\bar{k}$ is a number

$$p(\bar{k}) = \prod_{tt'\in\Im} g'(k(t),k(t')), \qquad (4)$$

where $\Im$ is a set of all neighboring pixel pairs, $g'(k,k')=c_=$ if $k=k'$, $g'(k,k')=c_{\neq}$ if $k\neq k'$ and $c_= > c_{\neq}$. Then the problem (2), which has not taken into account reasonable considerations (4), can be transformed into looking for a labeling with the highest a posteriori probability,

$$\begin{aligned} \bar{k}^* &= \arg\max_{\bar{k}\in K^T} p(\bar{k})\cdot p(\bar{x}/k) = \\ &= \arg\max_{\bar{k}\in K^T} \prod_{tt'\in\Im} g'(k(t),k(t'))\times \\ &\quad \times \prod_{t\in T} q'(k(t),x(t)), \end{aligned} \qquad (5)$$

where $q'(k(t),x(t))$ is a short designation for t-th multiplier in (1). The problem (5) can be represented in an equivalent form

$$\bar{k}^* = \arg\max_{\bar{k}\in K^T} \left[\sum_{tt'\in\Im} g(k(t),k(t')) + \sum_{t\in T} q_t(k(t)) \right], \quad (6)$$

where $g(k,k')=1$ if $k=k'$, $g(k,k')=0$ if $k\neq k'$, and $q_t(k) = a\ \log\ q'_t(k) + b$ with some values a and b. For a long time an exact solution to the problem (6) was unknown. An essential progress in this area has been achieved at the end of 90-th [4]. It was shown in [4] that in the considered case when ideal image is a binary one the optimization problem (6) can be reduced to max-flow problem and, consequently, is polynomially solvable at arbitrary numbers $q_t(k)$. However, if the ideal image is of the form $\bar{k}: T\to K$, $|K|>2$, not $\bar{k}: T\to\{0,1\}$, a set of all possible problems of the form (6) forms the *NP*-complete class. Consequently, it is hardly possible to solve them with an algorithm of polynomial complexity. So, one can see that visual analysis problems are difficult not only in a sense that there are lots of problems, which are unsolved yet. They are difficult in an exact computational meaning of the word. It applies equally both for huge practical projects with generally acknowledged significance and for special problems which seeming simplicity is extremely delusive. The considered example is just one of them.

A visual analysis needs the most powerful modern methods of computational optimization. Particularly, it requires solution to the problems that this paper is devoted to.

2. Main concepts and a problem formulation

Let T and K be two finite sets called set of objects and set of labels correspondingly. Let $\bar{k}: T\to K$ be a function called labeling, $k(t)$ be a value of this function for an object $t\in T$, K^T be a set of all possible labelings. The function $\bar{k}: T \to K$ will be also called a strict labeling to distinguish it from relaxed labeling defined below.

Let a subset $N(t) \subset T$ be defined for each object $t \in T$, which elements are called neighbors of the object t. The sets $N(t)$ are such that $t \notin N(t)$ and $t' \in N(t) \Leftrightarrow t \in N(t')$. Let symbol $\Im$ designate a set $\{\{t,t'\} \mid t \in T,\ t' \in N(t)\}$. So, designation $\{t,t'\} \in \Im$ is a short expression for $(t \in N(t'))\&$ $\&(t' \in N(t))$. Instead of a designation $\{t,t'\} \in \Im$ even shorter designation $tt' \in \Im$ will be used below.

An ordered object-label pair (k,t), $k \in K$, $t \in T$, will be called a vertex, unordered vertex pair $((k,t),(k',t'))$, such that $tt' \in \Im$, will be called an edge. We will say that a vertex (k^*,t^*) belongs to labeling $\bar{k}: T \to K$ if $k(t^*) = k^*$. We will say that an edge $((k,t),(k',t'))$ belongs to labeling $\bar{k}: T \to K$ if both the vertex (k,t) and the vertex (k',t') belong to it.

Let for each vertex (k,t) a number $q(t,k)$ be specified called its quality, as well as the quality $g((k,t),(k',t'))$ be specified for each edge $((k,t),$ $(k',t'))$. A quality $G(\bar{k})$ of a labeling $\bar{k}: T \to K$, $\bar{k} \in K^T$, is a total sum of qualities of vertices and edges that belong to the labeling,

$$G(\bar{k}) = \sum_{tt' \in \Im} g((t,k(t)),(t',k(t'))) + \sum_{t \in T} q(t,k). \quad (7)$$

An optimal (strict) labeling problem consists in looking for the best (strict) labeling

$$\bar{k}^* = \arg\max_{\bar{k} \in K^T} G(\bar{k}). \quad (8)$$

A problem set of such type forms an *NP*-complete class.

A quality array $(q(t,k), t \in T, k \in K)$, will be denoted shortly q, an array $(g((t,k),(t',k')),$ $tt' \in \Im,\ k \in K, k' \in K)$, will be denoted g, a pair (q, g) will be called a quality function.

Problems that we call relaxed labeling optimization are essentially simpler than the problem (7), (8). They are based on the following concepts. Let $\alpha(t, k)$, $t \in T$, $k \in K$, be a number, which is called a weight of a vertex (t, k), $t \in T$, $k \in K$, and $\beta((t, k), (t', k'))$, $tt' \in \Im$, $k \in K$, $k' \in K$, be a weight of an edge $((t, k), (t', k'))$. Let us denote α and β a vertex weights array $(\alpha(t,k) \mid t \in T,\ k \in K)$ and an edge weights array $(\beta((t, k), (t', k')) \mid tt' \in \Im, k \in K,$ $k' \in K)$ correspondingly. A pair (α, β) will be referred to as a weight function. A weight function (α, β) will be called a relaxed labeling if it satisfies the conditions

$$\begin{cases} \alpha(t,k) = \sum_{k' \in K} \beta((t,k),(t',k')), \\ \qquad\qquad t \in T, k \in K,\ t' \in N(t); & (9) \\ \sum_{k \in K} \alpha(t,k) = 1, \qquad t \in T; & (10) \\ \alpha(t,k) \ge 0, \qquad t \in T,\ k \in K; & (11) \\ \beta((t,k),(t',k')) \ge 0, \qquad tt' \in \Im,\ k \in K, \\ \qquad\qquad k' \in K. & (12) \end{cases}$$

The set of all possible solutions to the equatlity system (9)–(12) will be denoted A. For each relaxed labelling $(\alpha,\beta) \in A$ its quality

$$\begin{aligned} G(\alpha,\beta) = \sum_{tt' \in \Im} \sum_{k \in K} \sum_{k' \in K} \beta((t,k),(t',k')) \times \\ \times g((t,k),(t',k')) + \sum_{t \in T} \sum_{k \in K} \alpha(t,k) \cdot q(t,k) \end{aligned} \quad (13)$$

is defined or more shortly

$$G(\alpha,\beta) = \langle \alpha, q \rangle + \langle \beta, g \rangle, \quad (14)$$

where $\langle \bullet,\bullet \rangle$ means an inner product of two functions defined on the same domain. Relaxed labelling problem consists in looking for relaxed labelling (α^*,β^*) with the best quality,

$$G(\alpha^*,\beta^*) = \max_{(\alpha,\beta) \in A} G(\alpha,\beta). \quad (15)$$

A set of all possible problems of such type forms a certain subclass of linear programming problems and so is solvable in polynomial time. However, no practically good algorithm is known now for solution to all problems of the form (9)–(15).

A relaxed labeling may be frequently used as a rather acceptable substitute for a strict labeling due to the following reasons:

a) several subclasses of a strict labeling problem can be reduced to relaxed labeling; those are problems with acyclic neighborhood and problems with supermodular quality function [22]; in this paper we will add to this list a so-called permutated supermodular problems [23], which include all supermodular problems as well as all submodular problems with bipartite neighbourhood;

b) the quality of the best relaxed labeling is an upper estimate for the quality of the best strict labeling and so is a tool for final testing, whether some heuristically found strict labeling differs essentially from the best possible one or not;

c) the relaxed labeling is a component for discriminative learning of strict labeling algorithms [24];

d) at last, in spite of the fact that the set of all relaxed labeling problems forms a polynomially solvable class no practically good algorithm for their solution is known so far and it is a rather good motivation for looking for such algorithms.

3. The Eequivalent and trivial labeling problems

Input data both for strict and for relaxed labeling are the same. They are presented with a five-tuple $z=\langle T,\ K,\ \Im,\ q,\ g\rangle$, where T and K are two finite sets, $\Im$ is a subset of pairs of the form $\{t,\ t'\}$, $t \in T$, $t' \in T$, $t \neq t'$, q is a number array ($q(t, k)$, $t \in T$, $k \in K$), g is a number array ($g((t, k), (t', k'))$, $tt' \in \Im$, $k \in K$, $k' \in K$). Input data $z=\langle T, K, \Im,\ q,\ g\rangle$ will be sometimes called merely a problem, with no specification, whether a strict labeling is considered or relaxed one.

The following condition is evidently a sufficient condition of strict labeling optimality. If for labeling $\bar{k}^*: T \to K$ the conditions

$$q\left(t,k^*(t)\right)=\max_{k\in K} q(t,k),$$

$$g\left(\left(t,k^*(t)\right),\left(t',k^*(t')\right)\right)=\max_{k\in K,k'\in K} g\left((t,k),(t',k')\right),$$

are valid for each $t \in T$ and for each $tt' \in \Im$ then inequality $G\left(\bar{k}^*\right) \geq G\left(\bar{k}\right)$ is valid for each labeling $\bar{k} \in K^T$. This condition is too strong and, consequently, is trivial. So, the strict labeling problem will be called trivial if at least one optimal labeling satisfies above-mentioned sufficient optimality condition.

Similarly a trivial sufficient condition of the relaxed labeling optimality can be formulated. If for the relaxed labelling $\left(\alpha^*,\beta^*\right)\in A$ the conditions

$$q(t,k)<\max_{l\in K} q(t,l) \Rightarrow \alpha^*(t,k)=0,$$

$$g\left((t,k),(t',k')\right)<\max_{l\in K}\max_{l'\in K} g\left((t,l),(t',l')\right) \Rightarrow$$

$$\Rightarrow \beta^*\left((t,k),(t',k')\right)=0$$

are fulfilled for each vertex (k,t) and each edge $\left((k,t),(k',t')\right)$ then the inequality $G\left(\alpha^*,\beta^*\right) \geq$ $\geq G(\alpha,\beta)$ is valid for each relaxed labeling $(\alpha,\beta)\in A$.

Two strict labeling problems $z^1 =$ $=\langle T,\ K,\ \Im,\ g^1,\ q^1\rangle$ and $z^2=\langle T,\ K,\ \Im,\ g^2,\ q^2\rangle$ are called equivalent if the equality

$$\sum_{tt'\in\Im} g^1\left(\left(t,k(t)\right),\left(t',k(t')\right)\right)+\sum_{t\in T} q^1\left(t,k(t)\right)=$$

$$=\sum_{tt'\in\Im} g^2\left(\left(t,k(t)\right),\left(t',k(t')\right)\right)+\sum_{t\in T} q^2\left(t,k(t)\right)$$

is valid for each labeling. The following theorem has been proved in [22].

Theorem 1. Two strict labeling problems $z^1 =$ $=\langle T,\ K,\ \Im,\ g^1,\ q^1\rangle$ and $z^2=\langle T,\ K,\ \Im,\ g^2,\ q^2\rangle$ are equivalent if and only if there exist the numbers $\varphi_{tt'}(k)$, $t\in T$, $t'\in N(t)$, $k\in K$, which satisfy equalities

$$g^2\left((t,k),(t',k')\right)=g^1\left((t,k),(t',k')\right)+$$

$$+\varphi_{tt'}(k)+\varphi_{t't}(k'),\ tt'\in\Im,\ k\in K,\ k'\in K,$$

$$q^2(t,k)=q^1(t,k)-\sum_{t'\in N(t)}\varphi_{tt'}(k),\ t\in T,\ k\in K.$$

Numbers $\varphi_{tt'}(k)$, $t\in T$, $t'\in N(t)$, $k\in K$, in the theorem formulation are called potentials.

The theorem may be generalized for relaxed labeling.

Theorem 2. Let $\langle T, K, \Im, g^1, q^1 \rangle$ and $\langle T, K, \Im, g^2, q^2 \rangle$ be two labeling problems. The equality

$$\langle \alpha, q^1 \rangle + \langle \beta, g^1 \rangle = \langle \alpha, q^2 \rangle + \langle \beta, g^2 \rangle \qquad (16)$$

holds for each relaxed labeling $(\alpha, \beta) \in A$ if and only if such potentials $\varphi_{tt'}(k)$, $t \in T$, $t' \in N(t)$, $k \in K$, exist that

$$g^2\left((t,k),(t',k')\right) = \\ = g^1\left((t,k),(t',k')\right) + \varphi_{tt'}(k) + \varphi_{t't}(k'), \\ tt' \in \Im,\ k \in K,\ k' \in K, \qquad (17)$$

$$q^2(t,k) = q^1(t,k) - \sum_{t' \in N(t)} \varphi_{tt'}(k),\ t \in T,\ k \in K. \qquad (18)$$

Proof. Let us prove that if the condition (16) is valid then the conditions (17) and (18) are valid too. As the equality (16) is valid for each relaxed labeling it is valid also for relaxed labelings with integer weights $\alpha(t,k)$ and $\beta\left((t,k),(t',k')\right)$, i.e. for each strict labeling. Due to the theorem 1 the conditions (17), (18) are satisfied.

Let us prove that the conditions (17) and (18) imply the equality (16). Let us choose an arbitrary relaxed labelling $(\alpha, \beta) \in A$ and fix it for subsequent considerations. A weight $\beta\left((t,k),(t',k')\right)$ is defined for an edge, i.e. for unordered pair $\left((t,k),(t',k')\right)$, so

$$\beta\left((t,k),(t',k')\right) = \beta\left((t',k'),(t,k)\right). \qquad (19)$$

For each array of potentials $\varphi_{tt'}(k)$, $t \in T$, $t' \in N(t)$, $k \in K$, the equality

$$\sum_{tt' \in \Im} \sum_{k \in K} \sum_{k' \in K} \beta\left((t,k),(t',k')\right) \cdot \left[\varphi_{tt'}(k) + \varphi_{t't}(k')\right] = \\ = \sum_{t \in T} \sum_{t' \in N(t)} \sum_{k} \sum_{k'} \beta\left((t,k),(t',k')\right) \cdot \varphi_{tt'}(k) \qquad (20)$$

is valid. Indeed, let t_1 and t_2 be two neighbors, $t_1 t_2 \in \Im$, i.e. $t_2 \in N(t_1)$, $t_1 \in N(t_2)$. This pair is represented with a sum

$$\sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \left[\varphi_{t_1 t_2}(k) + \varphi_{t_2 t_1}(k')\right] \qquad (21)$$

at the left side of (20) and with a sum

$$\sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \varphi_{t_1 t_2}(k) + \\ + \sum_{k \in K} \sum_{k' \in K} \beta\left((t_2,k),(t_1,k')\right) \cdot \varphi_{t_2 t_1}(k) \qquad (22)$$

at the right-hand side of (20). The following chain is valid:

$$\sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \varphi_{t_1 t_2}(k) + \\ + \sum_{k \in K} \sum_{k' \in K} \beta\left((t_2,k),(t_1,k')\right) \cdot \varphi_{t_2 t_1}(k) = \\ = \sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \varphi_{t_1 t_2}(k) + \\ + \sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k'),(t_2,k)\right) \cdot \varphi_{t_2 t_1}(k) = \\ = \sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \varphi_{t_1 t_2}(k) + \\ + \sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \varphi_{t_2 t_1}(k') = \\ = \sum_{k \in K} \sum_{k' \in K} \beta\left((t_1,k),(t_2,k')\right) \cdot \left[\varphi_{t_1 t_2}(k) + \varphi_{t_2 t_1}(k')\right].$$

So, the numbers (21) and (22) are equal and, consequently, the equality (20) is valid.

For relaxed labelling (α, β) the equality

$$\alpha(t,k) = \sum_{k' \in K} \beta\left((t,k),(t',k')\right)$$

holds for each $t \in T$, $k \in K$, $t' \in N(t)$. That is why the equality (20) may be rewritten in the form

$$\sum_{tt' \in \Im} \sum_{k \in K} \sum_{k' \in K} \beta\left((t,k),(t',k')\right) \cdot \left[\varphi_{tt'}(k) + \varphi_{t't}(k')\right] = \\ = \sum_{t \in T} \sum_{k \in K} \alpha(t,k) \cdot \sum_{t' \in N(t)} \varphi_{tt'}(k). \qquad (23)$$

Due to the conditions (17) and (18) a difference between qualities $\langle \alpha, q^1 \rangle + \langle \beta, g^1 \rangle$ and $\langle \alpha, q^2 \rangle + \langle \beta, g^2 \rangle$ of the labelling (α, β) is

$$\sum_{tt' \in \Im} \sum_{k \in K} \sum_{k' \in K} \beta_{tt'}(k,k') \cdot \left[\varphi_{tt'}(k) + \varphi_{t't}(k')\right] - \\ - \sum_{t \in T} \sum_{k \in K} \alpha(t,k) \cdot \sum_{t' \in N(t)} \varphi_{tt'}(k).$$

Due to (23) this number is zero. The t**heorem is proved.**

Let us define for a quality function (q,g) its characteristic

$$P(q,g)=\sum_{tt'\in\Im}\max_{k\in K,k'\in K} g\left((t,k),(t',k')\right)+ +\sum_{t\in T}\max_{k\in K} q(t,k)$$

and let us call it a power of the quality function (q,g). Let a quality function (q,g) be transformed equivalently into the function (q',g') with potentials $\varphi_{tt'}(k)$, $t\in T$, $t'\in N(t)$, $k\in K$, so that

$$g'\left((t,k),(t',k')\right)=g\left((t,k),(t',k')\right)+\varphi_{tt'}(k)+\varphi_{t't}(k'),$$
$$t\in T,\ t'\in N(t),\ k\in K,\ k'\in K;$$
$$q'(t,k)=q(t,k)-\sum_{t'\in N(t)}\varphi_{tt'}(k), \qquad t\in T,\ k\in K.$$

In this case the power of the transformed quality function may be expressed explicitly via potentials so that

$$P(q',g')=\sum_{tt'\in\Im}\max_{k\in K,k'\in K}\Big[g\left((t,k),(t',k')\right)+ +\varphi_{tt'}(k)+\varphi_{t't}(k')\Big]+\sum_{t\in T}\max_{k\in K}\left[q(t,k)-\sum_{t'\in N(t)}\varphi_{tt'}(k)\right].$$

One can see that the power of a quality function depends convexly on potentials. As we will see, it is important because it shows a way for solution to wide subclasses of strict labeling problems. The following three theorems [22] show how it occurs.

Theorem 3. Let (q,g) be a quality function and *Z* be its equivalency class. If the function has a trivial equivalent then each function (q^*,g^*), which minimizes power in the class *Z*, is trivial.

Theorem 4. If the neighborhood $\Im$ contains no cycle on the set *T* then each strict labeling problem $\langle T,K,\Im,q,g\rangle$ has a trivial equivalent $\langle T,K,\Im,q^*,g^*\rangle$.

Let the label set *K* be an ordered set and qualities $g\left((t,k),(t',k')\right)$ satisfy the inequality

$$g\left((t,k_1),(t',k_2')\right)+g\left((t,k_2),(t',k_1')\right)\le$$
$$\le g\left((t,k_1),(t',k_1')\right)+g\left((t,k_2),(t',k_2')\right)$$

for each pair $tt'\in\Im$ and each quadruple $k_1\ge k_2$, $k_1'\ge k_2'$ of labels. The problem with such edge qualities is called supermodular.

Theorem 5. Each supermodular problem has a trivial equivalent.

The theorem can be generalized for the wider class of problems called permuted supermodular, which are defined in the following way [23] Let *I* be some ordered set and $i_t: K\to I$, $t\in T$, be a function, which defines numbering of labels, its own for each object $t\in T$. The problem $\langle T,K,\Im,q,g\rangle$ is called permuted supermodular if such numberings $i_t: K\to I$, $t\in T$, exist that inequality

$$g\left((t,k_1),(t',k_2')\right)+g\left((t,k_2),(t',k_1')\right)\ge$$
$$\ge g\left((t,k_1),(t',k_1')\right)+g\left((t,k_2),(t',k_2')\right)$$

is valid for each neighbor pair $tt'\in\Im$ and each label quadruple k_1,k_2,k_1',k_2' such that $i_t(k_1)\ge i_t(k_2)$, $i_{t'}(k_1')\ge i_{t'}(k_2')$.

Theorem 6. Each permuted supermodular problem has a trivial equivalent.

We will not adduce a proof of the theorem because more fine properties of permuted supermodularity will be analyzed below. The following theorem is almost evident.

Theorem 7. A power of trivial strict labelling problem equals to the quality of the best strict labeling.

Proof. Let $\bar{k}^*: T\to K$ be a labeling that implies a triviality of the problem $\langle T,K,\Im,q,g\rangle$. For the labelling $\bar{k}^*$ and any other labeling $\bar{k}: T\to K$ it is valid that

$$G(\bar{k}^*)=\sum_{tt'\in\Im} g\left((t,k^*(t)),(t',k^*(t'))\right)+ +\sum_{t\in T} q(t,k^*(t))=\sum_{tt'\in\Im}\max_{k\in K}\max_{k'\in K} g\left((t,k),(t',k')\right)+ +\sum_{t\in T}\max_{k\in K} q(t,k)\ge\sum_{tt'\in\Im} g\left((t,k(t)),(t',k(t'))\right)+ +\sum_{t\in T} q(t,k(t))=G(\bar{k}).$$

The theorem is proved.

Theorem 8. If the problem $\langle T,K,\Im,q,g\rangle$ minimizes the power in its equivalency class then its power equals a quality of the best relaxed labeling.

Proof. 1. Let P^* be a power of the quality function (g, q), which the theorem tells about,

$$P^* = \sum_{tt'\in\Im} \max_{k\in K,k'\in K} g\left(\left(t,k\right),\left(t',k'\right)\right) + \sum_{t\in T} \max_{k\in K} q\left(t,k\right), \quad (24)$$

and $P(\Phi)$ be a power of a quality function that is equivalent to (g, q) and is obtained with potentials $\Phi = \left(\varphi_{tt'}\left(k\right) \middle| \ t\in T,\ t'\in N\left(t\right),\ k\in K\right)$,

$$P\left(\Phi\right) = \sum_{tt'\in\Im} \max_{k\in K,k'\in K} \left[g\left(\left(t,k\right),\left(t',k'\right)\right) + \varphi_{tt'}\left(k\right) + \right.$$
$$\left. + \varphi_{t't}\left(k'\right) \right] + \sum_{t\in T} \max_{k\in K} \left[q\left(t,k\right) - \sum_{t'\in N(t)} \varphi_{tt'}\left(k\right) \right]. \quad (25)$$

Due to the theorem condition the inequality

$$P\left(\Phi\right) \ge P^* \quad (26)$$

holds for each array Φ of potentials.

2. Let (α, β) be an arbitrary relaxed labeling and

$$G\left(\alpha,\beta\right) = \sum_{tt'\in\Im} \sum_{k\in K,k'\in K} \beta\left(\left(t,k\right),\left(t',k'\right)\right) \times$$
$$\times g\left(\left(t,k\right),\left(t',k'\right)\right) + \sum_{t\in T} \sum_{k\in K} \alpha\left(t,k\right) \cdot q\left(t,k\right) \quad (27)$$

be its quality. The weights $\alpha(t, k)$ and $\beta((t, k), (t', k'))$ are non-negative, the equality $\sum_{k\in K} \alpha\left(t,k\right) = 1$ holds for each $k \in K$ and $\sum_{k\in K,k'\in K} \beta\left(\left(t,k\right),\left(t',k'\right)\right) = 1$ holds for each $tt' \in \Im$. It implies that each summand in the right-hand side of (24) is not less than the corresponding summand in the right-hand side of (27),

$$\max_{k\in K,k'\in K} g\left(\left(t,k\right),\left(t',k'\right)\right) \ge$$
$$\ge \sum_{k\in K,k'\in K} \beta\left(\left(t,k\right),\left(t',k'\right)\right) \cdot g\left(\left(t,k\right),\left(t',k'\right)\right),$$
$$\max_{k\in K,k'\in K} q\left(t,k\right) \ge \sum_{k\in K} \alpha\left(t,k\right) \cdot q\left(t,k\right).$$

Consequently, the inequality

$$P \ge G\left(\alpha,\beta\right) \quad (28)$$

is valid for each relaxed labeling $\left(\alpha,\beta\right)$.

3. Let us consider an array of weights $a\left(t,k\right)$, $k\in K$, $t\in T$, $b\left(\left(t,k\right),\left(t',k'\right)\right)$, which is not necessarily a relaxed labeling. However, they satisfy the restrictions

$$a\left(t,k\right) \ge 0,\ \sum_{k\in K} a\left(t,k\right) = 1,\ t\in T,\ k\in K, \quad (29)$$

$$b\left(\left(t,k\right),\left(t',k'\right)\right) \ge 0,\ \sum_{k\in K,k'\in K} b\left(\left(t,k\right),\left(t',k'\right)\right) = 1,$$
$$tt'\in\Im,\ k\in K,\ k'\in K, \quad (30)$$

$$g\left(\left(t,k\right),\left(t',k'\right)\right) < \max_{l\in K,l'\in K} g\left(\left(t,l\right),\left(t',l'\right)\right) \Rightarrow$$
$$\Rightarrow b\left(\left(t,k\right),\left(t',k'\right)\right) = 0, \quad (31)$$

$$q\left(t,k\right) < \max_{l\in K} q\left(t,l\right) \Rightarrow a\left(t,k\right) = 0. \quad (32)$$

Let us define a function

$$L\left(\Phi\right) = \sum_{tt'\in\Im} \sum_{k\in K,k'\in K} b\left(\left(t,k\right),\left(t',k'\right)\right) \times$$
$$\times \left[g\left(\left(t,k\right),\left(t',k'\right)\right) + \varphi_{tt'}\left(k\right) + \varphi_{t't}\left(k'\right) \right] +$$
$$+ \sum_{t\in T} \sum_{k\in K} \alpha\left(t,k\right) \cdot \left[q\left(t,k\right) - \sum_{t'\in N(t)} \varphi_{tt'}\left(k\right) \right],$$

which depends linearly on potentials $\varphi_{tt'}\left(k\right)$. Due to conditions (29)–(32) this function does not exceed the power $P\left(\Phi\right)$. So, the inequality

$$L\left(\Phi\right) \le P\left(\Phi\right) \quad (33)$$

is valid for any potentials and it becomes equality for zero potentials,

$$L\left(0\right) = P\left(0\right). \quad (34)$$

4. Due to (33), (34) a gradient of the linear function $L(\Phi)$ is a subgradient of the convex function $P(\Phi)$ at the zero point $\Phi = 0$. The gradient of the linear function $L(\Phi)$ is an array of numbers

$$\Delta\varphi_{tt'}\left(k\right) = a\left(t,k\right) - \sum_{t'\in N(t)} b\left(\left(t,k\right),\left(t',k'\right)\right),$$
$$t\in T,\ t'\in N\left(t\right),\ k\in K. \quad (35)$$

The same array obtained from numbers $a\left(t,k\right)$ and $b\left(\left(t,k\right),\left(t',k'\right)\right)$ that satisfy (29)–(32) is a subgradient of the convex function $P(\Phi)$. Even the stronger statement is valid that each subgradient of the function $P(\Phi)$ at the zero point $\Phi = 0$ has a form (35) for numbers $a\left(t,k\right)$ and $b\left(\left(t,k\right),\left(t',k'\right)\right)$ that satisfy (29)–(32).

5. By the theorem condition it is the zero point where the convex function $P(\Phi)$ takes its minimal value. Consequently, there exists a zero gradient at the zero point $\Phi = 0$, i.e. the numbers $a^0(t,k)$ and $b^0((t,k),(t',k'))$ that satisfy the conditions (29)–(32) as well as additional conditions

$$\Delta\varphi_{tt'}(k) = a^0(t,k) - \sum_{t' \in N(t)} b^0((t,k),(t',k')) = 0,$$

$$t \in T,\ t' \in N(t),\ k \in K.$$

It means that they form a relaxed labeling.

6. For this labeling due to conditions (29)–(32) the equality

$$\sum_{k \in K} a^0(t,k) \cdot q(t,k) = \max_{k \in K} q(t,k). \qquad (36)$$

holds for each $t \in T$ as well as the equality

$$\sum_{k \in K, k' \in K} b^0((t,k),(t',k')) \cdot g((t,k),(t',k')) = \\ = \max_{k \in K, k' \in K} g((t,k),(t',k')) \qquad (37)$$

holds for each $tt' \in \Im$. So, the following chain is valid:

$$G(a^0,b^0) = \sum_{tt' \in \Im} \sum_{k \in K, k' \in K} b^0((t,k),(t',k')) \times \\ \times g((t,k),(t',k')) + \sum_{t \in T} \sum_{k \in K} a^0(t,k) \cdot q(t,k) = \\ = \sum_{tt' \in \Im} \max_{k \in K, k' \in K} g((t,k),(t',k')) + \\ \sum_{t \in T} \max_{k \in K} q(t,k) = P^* \geq G(\alpha,\beta).$$

The first equality of the chain is valid due to the definition of the relaxed labeling quality. The second equality of the chain is valid due to the just proved equalities (36) and (37). The third equality is valid due to the definition of a quality function power. The inequality at the end of the chain is valid due to above-proved (28). So, the quality of the relaxed labeling (a^0, b^0) equals the power P^* and this labeling is not worse than any other relaxed labeling. **The theorem is proved.**

So, one can see that the transformation of the problem under solution into an equivalent problem with minimal power shows a general idea how to solve certain problem classes, which have been solved before in essentially different ways. Certainly, one has to acknowledge that the solution to the acyclic problem via power minimization is less effective than their solution with dynamic programming [25]. Similarly, it is more preferable to solve supermodular problems via their reduction to max-flow problems [4], not via power minimization. However, it is impossible to use or to modify dynamic programming ideas for solving the labeling problems with arbitrary neighborhood, not only acyclic. Similarly, max-flow method is inapplicable for arbitrary acyclic problems. As regards power minimization, it is a unified and general way for solution to all supermodular problems as well as all acyclic ones. Moreover, it will be shown later that the power minimization is a universal tool for all permuted supermodular problems [23], which can be solved neither with dynamic programming, nor with max-flow method. At last, power minimization is a universal way to compute the quality of the best relaxed labeling.

In spite of the whole attractiveness of the power minimization its main shortcoming consists in that no good algorithm is known for its implementation, only several separate attempts. Let us consider two attempts of the kind, their attractive and less attractive properties.

4. Diffusion and its formal property

Let $\langle T, K, \Im, q, g \rangle$ be input data for a labeling problem, either strict or relaxed one. These data can be equivalently transformed so that the weights of all vertices become zero. So, it will be assumed in this section that the labelling problem is defined with the quadruple $\langle T, K, \Im, g \rangle$ with zero values of all vertex weights. All subsequent equivalent transformations will be made so that weights of all vertices remain zero. It means that an equivalent transformation of the given problem $\langle T, K, \Im, g \rangle$ into the problem with minimal power consists in looking for potential values that minimize function of the form

$$P(\Phi) = \sum_{tt' \in \Im} \max_{k \in K, k' \in K} \Big[g((t,k),(t',k')) + \\ + \varphi_{tt'}(k) + \varphi_{t't}(k') \Big] \qquad (38)$$

under condition that the equality

$$\sum_{t'\in N(t)} \varphi_{tt'}(k) = 0 \tag{39}$$

holds for each vertex $t \in T$, $k \in K$.

A diffusion algorithm, known also as belief propagation, consists in sequential scanning of all objects $t \in T$ in an arbitrary order, sequential scanning of all vertices (t^*, k), $k \in K$, for each current object $t^* \in T$ and minimization of the power $P(\Phi)$ with respect only to those potentials $\varphi_{t^*t'}(k^*)$, $t' \in N(t^*)$, that relate to the current vertex (t^*, k^*). More exactly, diffusion algorithm or, simply, diffusion is an equivalent transformation of a quality function g into a function g' according to the following instructions:

for all $t \in T$ *and all* $k \in K$

{ *for all* $t' \in N(t)$

$$c(t') = \max_{k'\in K} g((t,k),(t',k'));$$

$$c = \frac{1}{|N(t')|} \sum_{t'\in N(t)} c(t');$$

for all $t' \in N(t)$ *and all* $k' \in K$

$$g(t,k,t',k') := g(t,k,t',k') + (c - c(t'));$$

}

Let us designate T a transformation of a quality function g with this algorithm so that $g' = T(g)$ means the transformation of the function g results in g'. The algorithm is an extremely simplified special case of the algorithms described in [26–28]. The considered special case has been analyzed in [29]. The result of the analysis likely holds for more general cases [26–28] .

Let us introduce additional concepts for formulation of the mentioned result. Let D be a set of all possible edges,

$$D = \{((t,k),(t',k')) \mid tt' \in \Im,\ k \in K,\ k' \in K\}.$$

A non-empty subset $D' \subset D$ of edges is called consistent if for each triple of objects t, t', t'' such that $t' \in N(t)$, $t'' \in N(t)$ and for each edge $((t,k), (t',k')) \in D'$ there exists an edge $((t,k),(t'',k'')) \in D'$. A quality function g is called ε-consistent if the set

$$\Big\{((t,k),(t',k')) \,\Big|\, g((t,k),(t',k')) \geq \geq \max_{l\in K,\, l'\in K} g((t,l),(t',l')) - \varepsilon\Big\}$$

contains a consistent subset. Each quality function g will be characterized by its inconsistency $\varepsilon(g)$, which is a minimal number ε that ensures its ε-consistency. A function g will be called consistent if $\varepsilon(g) = 0$. Inconsistency $\varepsilon(g)$ is a tractable characteristic of a quality function g.

The following theorem expresses the main property of diffusion [29].

Theorem 9. Let g^0 be a quality function, g^i, $i = 1, 2, \ldots, \ldots$ be a sequence of functions such that $g^i = T(g^{i-1})$, $i = 1, 2, \ldots$. In this case $\lim_{i\to\infty} \varepsilon(g_i) = 0$.

The theorem states that diffusion is a universal tool for equivalent transformation of an arbitrary quality function g into an ε-consistent equivalent for each positive ε. It will be shown how it allows to solve a certain subclass of strict labeling problems.

5. Permuted supermodular labeling problems

Let $\langle T, K, \Im, g\rangle$ be a labeling problem, either strict or relaxed, I be an ordered set, $i_t : K \to I$ be a label numbering defined for each object $t \in T$ and its own for each object. A quality function g will be called permuted supermodular if such numberings $i_t : K \to I$, $t \in T$, exist that for each pair $tt' \in \Im$ and for each quadruple k_1, k_2, k_1', k_2' such that $i_t(k_1) \geq i_t(k_2)$, $i_{t'}(k_1') \geq i_{t'}(k_2')$ the inequality

$$g((t,k_1),(t',k_2')) + g((t,k_2),(t',k_1')) \leq \leq g((t,k_1),(t',k_1')) + g((t,k_2),(t',k_2'))$$

holds. Certainly, the class of permuted supermodular functions is much wider than the class of supermodular functions. Particularly, if neighborhood $\Im$ forms a bipartite graph on a set T then each submodular function is permuted supermodular.

If for a permuted supermodular function numberings $i_t : K \to I$, $t \in T$, would be known the corresponding strict labeling problem could be reduced to a max-flow problem and solved with the known algorithms. The problem becomes more complex if the numberings $i_t : K \to I$, $t \in T$, are unknown and only their existence is ensured. A problem becomes even more complex if it is unknown whether the problem under solution is permuted supermodular or not. In this case it would be necessary, first of all, to recognize whether the problem is permuted supermodular and then either to solve it or not. Such way is possible if a quality of each edge does not equal ($-\infty$). It is known [23] that in this case it is possible to recognize permuted supermodularity in a polynomial time and to find corresponding numberings. However, if some edges can have a quality ($-\infty$) such way is not possible.

We will show that the diffusion can solve approximately (with an arbitrary small but non-zero error) and sometimes even exactly all permuted supermodular problems as well as many others. This statement holds not only for diffusion but for any other algorithm that transforms an arbitrary quality function into its ε-consistent equivalent.

A permuted supermodular quality function has the following properties.

Theorem 10. If g is a permuted supermodular quality function then each its equivalent g' is also permuted supermodular.

Proof. Due to the equivalency of g and g' such potentials $\varphi_{tt'}(k)$, $t \in T$, $t' \in N(t)$, $k \in K$, exist that

$$\begin{gathered} g'\big((t,k),(t',k')\big) = \\ g\big((t,k),(t',k')\big) + \varphi_{tt'}(k) + \varphi_{t't}(k'), \\ tt' \in \mathfrak{I},\ k \in K,\ k' \in K. \end{gathered} \tag{40}$$

Due to the permuted supermodularity of g such numberings $i_t : K \to I$, $t \in T$, exist that inequality

$$\begin{gathered} g\big((t,k_1),(t',k_2')\big) + g\big((t,k_2),(t',k_1')\big) \le \\ \le g\big((t,k_1),(t',k_1')\big) + g\big((t,k_2),(t',k_2')\big) \end{gathered} \tag{41}$$

holds for each $tt' \in \mathfrak{I}$ and each quadruple k_1, k_2, k_1', k_2' such that $i_t(k_1) \ge i_t(k_2)$, $i_{t'}\left(k_1'\right) \ge i_{t'}\left(k_2'\right)$. Due to (40) the same inequality is also valid for g'. Indeed,

$$\begin{aligned} & g'\big((t,k_1),(t',k_2')\big) + g'\big((t,k_2),(t',k_1')\big) = \\ & = \Big(g\big((t,k),(t',k')\big) + \varphi_{tt'}(k_1) + \varphi_{t't}(k_2')\Big) + \\ & + \Big(g\big((t,k_2),(t',k_1')\big) + \varphi_{tt'}(k_2) + \varphi_{t't}(k_1')\Big) \le \\ & \le \Big(g\big((t,k_1),(t',k_1')\big) + \varphi_{tt'}(k_1) + \varphi_{t't}(k_1')\Big) + \\ & + \Big(g\big((t,k_2),(t',k_2')\big) + \varphi_{tt'}(k_2) + \varphi_{t't}(k_2')\Big) = \\ & = g'\big((t,k_1),(t',k_1')\big) + g'\big((t,k_2),(t',k_2')\big). \end{aligned}$$

The theorem is proved.

So, each permuted supermodular problem can be represented as a permuted supermodular and ε-consistent problem and it is valid for arbitrary positive ε. Particularly, it can be made with diffusion. It is a way to find an almost best strict labeling and sometimes even an exactly best labeling.

Theorem 11. Let $\langle T, K, \mathfrak{I}, g \rangle$ be a problem with permuted supermodular ε-consistent quality function g with a power P. Then a strict labelling $\bar{k}^* : T \to K$ exists with quality

$$G\left(\bar{k}^*\right) \ge P - |\mathfrak{I}| \cdot 2\varepsilon. \tag{42}$$

Proof. Let us define a number

$$c(t,t') = \max_{k \in K} \max_{k' \in K} g\big((t,k),(t',k')\big) \tag{43}$$

for each $tt' \in \mathfrak{I}$ as well as an edge subset

$$\begin{gathered} D_\varepsilon = \Big\{\big((t,k),(t',k')\big) \mid g\big((t,k),(t',k')\big) \ge \\ \ge c(t,t') - \varepsilon\Big)\Big\} \end{gathered} \tag{44}$$

and its consistent subset $D_\varepsilon^* \subseteq D_\varepsilon$. Let us define the subset

$$K(t) = \Big\{k \in K \mid \exists k' \Big(\big((t,k),(t',k')\big) \in D_\varepsilon^*\Big)\Big\} \tag{45}$$

for each $t \in T$ and some $t' \in N(t)$. Due to consistency of the subset D_ε^* the definition (45) does not depend on what neighbor $t' \in N(t)$ is used at the right-hand side of (45) and due to the consistency of D_ε^* all subsets $K(t)$ are non-empty.

A labeling $\bar{k}^* : T \to K$, existence of which has to be proved, is the labeling

$$k^*(t) = \arg\max_{k \in K(t)} i_t(k).$$

Let us prove that for this labeling inequality (42) holds. Let us choose an arbitrary neighbor pair $tt' \in \Im$ and fix it for subsequent considerations. The set $K(t)$ contains a label k, for which

$$c(t,t') \ge g\left((t,k),(t',k^*(t'))\right) \ge$$
$$\ge c(t,t') - \varepsilon,\ i_t(k) \le i_t\left(k^*(t)\right),$$

and set $K(t')$ contains a label k', for which

$$c(t,t') \ge g\left((t,k^*(t)),(t',k')\right) \ge$$
$$\ge c(t,t') - \varepsilon,\ i_{t'}(k') \le i_{t'}\left(k^*(t')\right).$$

Moreover,

$$g\left((t,k),(t',k^*(t'))\right) \le c(t,t').$$

For the quadruple k, $k^*(t)$, k', $k^*(t')$ the inequality

$$g\left((t,k^*(t)),(t',k^*(t'))\right) + g\left((t,k),(t',k')\right) \ge$$
$$\ge g\left((t,k),(t',k^*(t'))\right) + g\left((t,k^*(t)),(t',k')\right)$$

is valid and, consequently, the weaker inequalities

$$g\left((t,k^*(t)),(t',k^*(t'))\right) + c(t,t') \ge$$
$$c(t,t') - \varepsilon + c(t,t') - \varepsilon,$$

$$g\left((t,k^*(t)),(t',k^*(t'))\right) \ge c(t,t') - 2\varepsilon$$

are valid too. The last inequality holds for each $tt' \in \Im$ and that is why the quality $G\left(\bar{k}^*\right)$ of the labelling $\bar{k}^*$ is

$$G\left(\bar{k}^*\right) = \sum_{tt' \in \Im} g\left((t,k^*(t)),(t',k^*(t'))\right) \ge$$
$$\ge \sum_{tt' \in \Im} c(t,t') - 2\varepsilon|\Im| = P - 2\varepsilon|\Im|.$$

The theorem is proved.

The theorem states an existence of a labeling with a quality that is almost optimal and sometimes is exactly optimal. However, the theorem says nothing how this labeling can be found when numberings $i_t : K \to I, t \in T$, are unknown. The following consideration shows how the labeling can be found.

If a permuted supermodular quality function g takes integer values then the quality of the best labeling can be defined exactly, not approximately. It is necessary to transform the given quality function into an ε-consistent function g' with $\varepsilon < \dfrac{1}{2|\Im|}$. In this case the quality of the best labeling is the greatest integer, which does not exceed $\sum_{tt' \in \Im} \max_{k \in K, k' \in K} g'\left((t,k),(t',k')\right)$.

We describe an algorithm that solves all permuted supermodular problems but bypasses a question whether the problem under solution is permuted supermodular or not. If the problem is permuted supermodular the algorithm returns a labeling, which is surely optimal. If some other problem is presented for a solution the algorithm either returns a labeling, which is surely optimal, or stops with a comment that presented problem is not permuted supermodular. So, the algorithm solves all permuted supermodular problems as well as many others and avoids recognition of their permuted supermodularity.

The algorithm is based on equivalent transformation of a quality function into ε-consistent, let us say, with diffusion. In addition, the algorithm makes additional non-equivalent transformation of a quality function, which we call fixation of a vertex. Let g be a quality function and (k^*, t^*) be some vertex. Its fixation consists in transformation of g into g' so that $g'\left((t^*,k),(t',k')\right) = -\infty$ if $k \ne k^*$, $t' \in N(t^*)$, $k \in K'$, and $g'\left((t,k),(t',k')\right) = g\left((t,k),(t',k')\right)$ for all other edges. Fixation of a vertex (k^*, t^*) excludes from subsequent processing all labellings that do not contain the vertex (k^*, t^*) and does not change qualities of all other labellings, which contain the vertex (k^*, t^*). If a quality function is permuted supermodular it remains permuted supermodular after fixation of any vertex.

Let $\langle T, K, \Im, g \rangle$ be a labeling problem, not necessarily permuted supermodular. It is assumed only that it takes integer values. The algorithm for a wide class of problems, which includes all permuted supermodular problems, consists in the following.

1. *Define a number* $\varepsilon < \frac{1}{2|\Im|}$;

   for each $t \in T$ *define a subset* $K(t) = K$;

   transform a function g into an equivalent ε-consistent function g;

   look for the greatest integer number c *not greater than* $\sum_{tt' \in \Im} \max_{k \in K, k' \in K} g((t,k),(t',k'))$.

Comment. Afterwards either the algorithm will find a labeling with quality c, and it will be the best labeling, or it will stop with a comment that the problem under solution is not permuted supermodular.

2. *Find an object* $t^* \in T$ with $|K(t^*)| > 1$;

   if there is no such object go to p.5.

3. *For each label* $k^* \in K(t^*)$

   { *fix the vertex* (k^*, t^*) *and save the result of fixation as* g';

   transform the function g' *into an equi valent ε-consistent function* g'';

   if $\sum_{tt' \in \Im} \max_{k \in K, k' \in K} g''((t,k),(t',k')) > c$

   % CONDITION%

   { $K(t^*) = \{k^*\}$; $g = g''$; *go to* p.2;}

   }

Comment. If the problem under solution is permuted supermodular the CONDITION will be satisfied at least for one label k^*. This condition can be satisfied even for some problems, which are not permuted supermodular.

4. Stop;

Comment. The algorithm stops here only if a problem under solution is not permuted supermodular.

5. Stop;

Comment. If the algorithm stops here it means successful ending. The labeling $\overline{k}^* : T \to K$ with values $k^*(t) \in K(t)$, $t \in T$, has a quality c and there is no labeling with better quality.

One can see that the diffusion has some nice and even unexpected properties as well as other algorithms which ensure equivalent transformation of a quality function into ε-consistent one. They enable to solve a wide range of strict labeling problems, not relaxed. However, they do not justify the hopes that they can solve all relaxed labeling problems, which are seemingly simpler than strict ones. They do not ensure equivalent transformation of the quality function into a function with a minimal power because the consistency of a function is only a necessary condition of power minimum, not sufficient. It means that the algorithms of such type will not find a trivial equivalent for some strict labeling problems even if such trivial equivalent exists. It is their weighty disadvantage. In the next section subgradient optimization of the problem power is described, which is free of just this disadvantage but has other unpleasant properties.

6. Subgradient minimization of a problem power

Let $\langle T, K, \Im, g, q \rangle$ be input data for a relaxed labeling problem. Theorem 8 states that calculating of the best relaxed labeling quality is reduced to looking for potentials $\varphi_{tt'}(k)$, $t \in T$, $t' \in N(t)$, $k \in K$, which minimize a power

$$P(\Phi) = \sum_{tt' \in \Im} \max_{k \in K, k' \in K} \Big[g((t,k),(t',k')) + \varphi_{tt'}(k) + \varphi_{t't}(k') \Big] + \sum_{t \in T} \max_{k \in K} \left[q(t,k) - \sum_{t' \in N(t)} \varphi_{tt'}(k) \right]. \quad (46)$$

As far as a power $P(\Phi)$ depends convexly on potentials its minimum can be found with subgradient descent [30]. In respect to our problem subgradient minimization consists in the following algorithm.

Initialization of the algorithm

1. *define the sequence of numbers* γ_i, $i = 1, 2, \ldots$, *so that* $\lim_{i \to \infty} \gamma_i = 0$, $\sum_{i=1}^{\infty} \gamma_i = \infty$;
2. *for each edge assign initial value* $g^0((t, k), (t', k'))$ *of its quality equal to quality* $g((t, k), (t', k'))$ *given in input data*;
3. *for each vertex assign initial value* $q^0(t, k)$ *of its quality equal to quality* $q(t, k)$ *given in input data*;
4. *assign* $i = 0$.

Repeated iteration of the algorithm

1. *assign zero values to all potentials* $\varphi_{tt'}(k)=0$, $t\in T$, $t'\in N(t)$, $k\in K$;
2. *for each object* $t\in T$ *and each object* $t'\in N(t)$ *choose any label* $k\in K$ *such that* $q^i(t,k)=\max_{l\in K} q^i(t,l)$ *and assign* $\varphi_{tt'}(k)=1$;
3. *for each pair* $tt'\in\Im$ *choose labels k and* k' *such that* $g^i((t,k),(t',k'))= = \max_{l\in K, l'\in K} g^i((t,l),(t',l'))$, *and do* $\varphi_{tt'}(k):= := \varphi_{tt'}(k)-1$; $\varphi_{t't}(k'):=\varphi_{t't}(k')-1$;
4. *for each vertex* $(t,k), t\in T$, $k\in K$, *calculate its new quality* $q^{i+1}(t,k):=q^i(t,l)- -\gamma_i \sum_{t'\in N(t)} \varphi_{tt'}(k)$;
5. *for each edge* $((t,k),(t',k'))$, $tt'\in\Im$, $k\in K$, $k\in K$, *calculate its new quality* $g^{i+1}((t,k),(t',k')):=g^i((t,k),(t',k'))+ +\gamma_i(\varphi_{tt'}(k)+\varphi_{t't}(k'))$;
6. *increment* $i:=i+1$ *and go to* p. 1.

For this algorithm the following statement holds that follows from the theory of subgradient descent [30]. Let (q^*, g^*) be a quality function that is equivalent to the initial function (q, g) and minimizes a power. Let P^* be a value of the minimum and let P^i be a power of a current function (q^i, g^i) obtained with the algorithm. In this case $\lim_{i\to\infty} P^i = P^*$. It is an important positive property of the subgradient descent compared with other algorithms, which ensure only obtaining a function with arbitrary small inconsistency. If a problem under solution does have a trivial equivalent then the subgradient descent approaches it and becomes arbitrarily near to it. However, the presented version of the subgradient descent has quite evident shortcomings, which follow rather from a gap in our knowledge of subgradient methods than from a disadvantage of the method itself.

First of all, there is no stop condition in the algorithm. Certainly, the convergence $\lim_{i\to\infty} P^i = P^*$ implies that for arbitrary small $\varepsilon>0$ the inequality $P^i - P^* < \varepsilon$ will be achieved in a finite time. However, it is unknown how to recognize that this condition is already satisfied if the value P^* is unknown.

Then, the general theory of subgradient descent imposes too weak restrictions on the sequence $\gamma_i, i=1,2,\ldots$. Of course, subgradient descent ensures a convergence to minimum at any sequence $\gamma_i, i=1,2,\ldots$, such that $\lim_{i\to\infty}\gamma_i=0$, $\sum_{i=1}^{\infty}\gamma_i=\infty$. However, this convergence is frequently too slow for practical use. Evidently, considerable efforts will be required yet to remove these imperfections.

7. Concluding remarks: where are good algorithms for the relaxed labeling problem?

Both diffusion and subgradient descent have a common feature that probably determines their common imperfections. They transform a quality function (q,g) though the main goal is to find the best relaxed labeling (α,β), not to represent a quality function in such or another convenient form. However, described algorithms do nothing with relaxed labeling. Relaxed labeling does not act in the algorithms at all. Future algorithms, hopefully free of mentioned shortcomings, should dispose some weight function (α,β) and some quality function (q,g) at each stage of their work and improve both of them step-by-step simultaneously. Moreover, a current weight function has not necessarily to be just a relaxed labeling with monotonously increasing quality. It can be a weight function that improves in some other sense. Let us show an example how it can occur.

The initial aim consisted in solving the following linear programming problem: for a given quality function (q,g) a weight function (α,β) had to be found that maximized a quality

$$\sum_{tt'\in\Im}\sum_{k\in K}\sum_{k'\in K}\beta((t,k),(t',k'))\cdot g((t,k),(t',k'))+ +\sum_{t\in T}\sum_{k\in K}\alpha(t,k)\cdot q(t,k) \tag{47}$$

under the conditions

$$\begin{cases} \alpha(t,k) = \sum_{k' \in K} \beta((t,k),(t',k')), \\ \qquad\qquad t \in T, k \in K,\ t' \in N(t); & (48) \\ \sum_{k \in K} \alpha(t,k) = 1, \qquad t \in T; & (49) \\ \beta((t,k),(t',k')) \geq 0,\ tt' \in \Im,\ k \in K, k' \in K. & (50) \end{cases}$$

A labeling (α, β) is evidently optimal if it satisfies restrictions (48), (49), (50) and additional conditions

$$\begin{cases} q'(t,k) < \max_{l \in K} q'(t,l) \Rightarrow \alpha(t,k) = 0, & (51) \\ g'((t,k),(t',k')) < \max_{l \in K} \max_{l' \in K} g'((t,l), \\ \qquad (t',l')) \Rightarrow \beta((t,k),(t',k')) = 0; & (52) \\ (q',g') \sim (q,g). & (53) \end{cases}$$

The condition (53) is a short designation of the functions equivalency for (q', g') and (q, g). A solution of the optimization problem (47)–(50) coincides with a solution of the relation system (48)–(53). An approximate solution to the last system can be found so that the condition (48) is initially relaxed and then gradually strengthened. An optimization problem (47)–(50) is reduced in such a way to another optimization problem, namely, to looking for such weight function (α, β), quality function (q, g) and the minimal value Δ^2 that the system of conditions

$$\begin{cases} \sum_{t \in T} \sum_{k \in K} \sum_{k' \in K} \left(\alpha(t,k) - \sum_{k' \in K} \beta((t,k),(t',k')) \right)^2 \leq \Delta^2; & (54) \\ q'(t,k) < \max_{l \in K} q'(t,l) \Rightarrow \alpha(t,k) = 0, t \in T, k \in K; & (55) \\ g'((t,k),(t',k')) < \max_{l \in K} \max_{l' \in K} g'((t,l),(t',l')) \Rightarrow \\ \qquad \Rightarrow \beta((t,k),(t',k')) = 0; & (56) \\ \sum_{k \in K} \alpha(t,k) = 1, \qquad t \in T; & (57) \\ \beta((t,k),(t',k')) \geq 0, \qquad tt' \in \Im,\ k \in K, k' \in K; & (58) \\ (q',g') \sim (q,g). & (59) \end{cases}$$

remains consistent. Such representation of the initial problem has the advantage that the minimal value of Δ^2 is known to be zero and only all other variables have to be specified so that they do not contradict this zero value. Consequently, if some algorithm would be available, which decreases Δ^2 step-by-step with its convergence to zero, no difficulty would arise with a stop condition. The algorithm should be stopped, when Δ^2 becomes small enough.

Relation system (54)–(59) has also another property that hopefully will result in an appropriate algorithm for converging Δ^2 to zero. Let the weight function (α, β), the quality function (q, g) and the number Δ^2 satisfy the relation system (54)–(59). In this case at least one of the following four statements is valid:

1. the weight function (α, β) is an optimal relaxed labeling;
2. there exist such labels k^* and k^{**} and an object t^* that the weights $\alpha(t^*, k^*)$ and $\alpha(t^*, k^{**})$ can be changed without violating conditions (54)–(59) and so that the left-hand side of (54) and, consequently, also a value of Δ^2 decrease;
3. there exists such an edge $((t, k), (t', k'))$ that the weight $\beta((t, k), (t', k'))$ can be changed without violating conditions (54)–(59) and so that the left-hand side of (54) and, consequently, also a value of Δ^2 decrease;
4. a quality function (q', g') can be changed without violating conditions (54)–(59) so that its power decreases.

Though exact solutions of the problems (47)–(50) and (54)–(59) coincide, their approximate solutions have a different nature. An approximate solution to the problem (47)–(50) means looking for a weight function such that it is a relaxed labeling. Though this labeling is not exactly optimal, its quality can become arbitrarily near to the best possible quality. An approximate solution to the problem (54)–(59) means that the best weight function is found. Though this function is not a relaxed labeling because of violating the conditions (48), this violation can be made arbitrarily small. So, practically good algorithms can be hopefully obtained if instead of an exact solution to initial problem (47)–(50) an approximate solution to other problem is pursued, with a possibility to bring the substitute problem arbitrary near to the initial one.

It is possible to move further and to relax other restrictions of the system (54)–(59) as well. For example, the conditions (55) and (56) can be substituted with weaker requirement to ensure small value of the "difference"

$$\sum_{t\in T}\left(\max_{k\in K} q'(t,k)-\sum_{k\in K}\alpha(t,k)\cdot q'(t,k)\right)+$$
$$+\sum_{tt'\in\Im}\left(\max_{k\in K,k'\in K} g'\left((t,k),(t',k')\right)-\right.$$
$$\left.-\sum_{k\in K}\sum_{k'\in K}\beta\left((t,k),(t',k')\right)\cdot g'\left((t,k),(t',k')\right)\right)$$

with subsequent minimization of the function

$$F(\alpha,\beta,q,g)=$$
$$=\sum_{t\in T}\sum_{k\in K}\sum_{k'\in K}\left(\alpha(t,k)-\sum_{k'\in K}\beta\left((t,k),(t',k')\right)\right)^2+$$
$$+\sum_{t\in T}\left(\max_{k\in K} q'(t,k)-\sum_{k\in K}\alpha(t,k)\cdot q'(t,k)\right)+ \quad (60)$$
$$+\sum_{tt'\in\Im}\left(\max_{k\in K,k'\in K} g'\left((t,k),(t',k')\right)-\right.$$
$$\left.-\sum_{k\in K}\sum_{k'\in K}\beta\left((t,k),(t',k')\right)\cdot g'\left((t,k),(t',k')\right)\right),$$

under conditions

$$\begin{cases}\sum_{k\in K}\alpha(t,k)=1, & t\in T; \quad (61)\\ \beta\left((t,k),(t',k')\right)\ge 0,\ tt'\in\Im,\ k\in K, k'\in K; & (62)\\ (q',g')\sim(q,g). & (63)\end{cases}$$

An exact solution to this problem coincides with an exact solution to the initial problem of the optimal relaxed labeling. However, its representation in the form (60)–(63) has an advantage because it is known in this representation that the minimal value of the function under minimization is zero. An approximate solution to the problem (60)–(63) may occur to be an appropriate substitute for the initial problem.

Formal analysis of the problems (54)–(59) and (60)–(63) will be shown in next publications.

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